

MATHEMATICAL MODELING OF LUBRICATION FLOWS IN THIN FILM HYDRODYNAMICS

Mehul Tithlia

Asst. Prof. (Mathematics)

Dolat –Usha Institute of Applied Sciences and Dhiru-Sarla Institute of Management & Commerce, Valsad,
Gujarat

Abstract

Thin film hydrodynamics is a basic component of mechanical and industrial systems with lubrication being central in reducing friction, wear and energy loss. Universal behavior of thin fluid films determines the behaviour of bearings, seals, sliders, gears, coating processes and microfluidic devices and is highly important in theoretical understanding as well as in practical design. Lubrication flows are mathematical modelling to offer a systemic approach to study the key parameters like the pressure distribution, load carrying capacity, velocity profiles and shear stress that forms in thin films under different working conditions. In this work, the theory of lubrication is formulated in terms of a detailed derivation of the governing equations on the basis of the NavierStokes equations which are further simplified into thin-film equations and further simplified to form the classical Reynolds equation. An analytical and numerical solutions methods are also investigated to prove their usefulness in the prediction of flow behavior in both simple and complicated geometries. The paper also discusses the effect of properties of fluids, surface geometry and conditions and the effect of these models on the lubrication performance in terms of the ability to maximize mechanical efficiency, reliability and inform the design of more sophisticated tribological systems.

Keywords: Thin film hydrodynamics, lubrication modeling, load-carrying capacity, pressure distribution, shear stress, mechanical bearings.

Introduction:

The lubrication theory is a pioneering area in fluid mechanics and tribology, concerned with the behavior of viscous fluids trapped in geometrically constrained mediums in which one of the characteristic dimensions (the film thickness) is considerably smaller than the rest. Such thin fluid films are ubiquitous in a broad field of applications in engineering and technologies, including journal and thrust bearings, mechanical seals, sliders, gears and biomedical devices such as artificial joints and microfluidic channels [1]. Lubrication is primarily aimed at minimizing friction and wear between moving solid surfaces by maintaining a constant fluid film, thus maximizing the mechanical performance and efficiency and the life span of engineering components. Lubrication lowers the energy wastage, decreases thermal stress, and alleviates the degradation of materials by avoiding direct contact between surfaces.

In hydrodynamic lubrication regime, a thin film of lubricant exists between the interacting surfaces and pressure is created because of the relative movement of the surfaces and the geometry of the surfaces [2]. This pressure accumulation allows the film of fluid to be able to take external loads and in effect sustain the interacting surfaces without physical contact. The fluid film thickness is usually many orders of magnitude less than the characteristic dimension of the contact surfaces and therefore the flow is very delicate to changes in surface geometry, motion velocity, fluid properties and operating conditions. This sensitivity implies that design and control in engineering must be carefully designed and controlled in areas where high speed machinery, precision bearings, and micro- or nano-scale devices [3].

The complex thin film lubrication flow behavior can be analysed and predicted well using mathematical modelling [4]. Using governing equations of fluid mechanics, such as the Navier-Stokes equations, and making suitable assumptions about thin films makes it possible to find out important parameters such as pressure distribution, velocity and flow field, shear stresses, changes in film thickness, and loading capacity. These forecasts play a major role in the choice of the right lubricants, bearing design, reduction of the frictional losses, and elimination of surface wear [5].

This paper aims at establishing a mathematical model of thin film lubrication where simplified equations that are based on thin film assumptions will be derived systematically. Derivation and analysis of classical Reynolds equation which is the foundation of hydrodynamic lubrication theory are introduced in particular [6]. The two methods of solving lubrication equations are also discussed showing how they can be applied in real-life engineering context of simple and complex geometries, steady and unsteady flow, and Newtonian and non-Newtonian fluids. Having given a simple yet a thorough insight in the field of thin film hydrodynamics, the work is expected to assist in the efficient design, optimization and functioning of contemporary mechanical systems, including standard bearings to complex microfluidic and bio medical systems [7].

Hydrodynamic Lubrication and Thin Film Flows

Under the hydrodynamic lubrication regime, the thin layer of lubricant that exists between the interacting surfaces creates pressure which is a result of the relative movement of the surfaces in addition to the geometric arrangement of the surfaces [8]. This accumulation of pressure allows the fluid film to sustain external loads, which essentially hold the interacting surfaces apart, having no physical contact. The fluid film thickness normally differs by orders of magnitude with the characteristic dimensions of the contact surfaces as such the flow is highly sensitive to distinctions in surface geometry, velocity of motion, fluid properties and operating conditions. This kind of sensitivity requires extreme design and control during engineering work, especially in high-speed equipment, fine bearings and micro- or nano-scale systems [9].

The complex behavior of thin film lubrication flows can be analysed and predicted by mathematical modelling to give a strong framework [10]. Through the use of governing equations of fluid mechanics, e.g. NavierStokes equations and the application of assumptions used in thin films it is possible to calculate such critical parameters as pressure distribution, velocity and flow profiles, shear stresses, change in film thickness, and resultant loadcarrying capacity. Such forecasts are useful in the choice of appropriate lubricants, optimization of bearings in design, reduction of frictional losses and in avoiding surface wear [11].

This paper is aimed at the derivation of a mathematical model of thin film lubrication through the derivation of simplified equations on the assumption of a thin film [12]. Specifically, the derivation and analysis of classical Reynolds equation are provided, since it is the basis of the hydrodynamic lubrication theory [13]. Analytical and numerical methods of solving lubrication equations are described and examples of their application to practical engineering problems of simple and non-simple geometry, steady and unsteady flow and in Newtonian and non-Newtonian fluids are given [14]. This work will enhance the design, optimization, and functioning of the present-day mechanical systems, both the traditional bearings and the high-tech microfluidic and biomedical devices, by offering a complete insight into the mechanism of how thin films work [15].

Mathematical Formulation:

Mathematical formulation gives a structured approach to explain the characteristic of the thin film lubrication flows and to determine the key parameters like distribution of pressure, shear stress, velocity

profile, and load carrying capacity. Beginning with the basic principles of fluid mechanics, the governing equations may be simplified on the basis of proper assumptions made about thin films, including inertial effects, film thickness, and bearing dimensions being small, and the laminar flow regime. The method enables the researchers and engineers to obtain the models that are useful in capturing the dynamics of the fluid film without making them computationally infeasible in both analytical and numerical investigations.

Governing Equations

The flow of a Newtonian viscous incompressible fluid in thin film lubrication is governed by the Navier–Stokes equations and the continuity equation:

Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad (1)$$

Thin Film Assumptions

In lubrication problems, the film thickness h is much smaller than the characteristic length L of the surfaces, i.e., $h \ll L$. Under this condition, the following simplifications are valid:

- Pressure variations across the film thickness are negligible ($\partial p / \partial z \approx 0$)
- Inertial forces are small compared to viscous forces, allowing for the neglect of convective terms.
- Flow is laminar and dominated by viscous shear.
- No-slip boundary conditions are usually assumed at solid surfaces.

These assumptions reduce the Navier–Stokes equations to a form suitable for thin film flow analysis.

Velocity Profile in Thin Films

For a Newtonian fluid between a stationary lower surface and a moving upper surface with velocity U , the simplified momentum equation in the x -direction is:

$$\mu \frac{\partial^2 u}{\partial z^2} = \frac{\partial p}{\partial x} \quad (2)$$

Applying boundary conditions:

- $u(0) = 0$ (stationary lower surface)
- $u(h) = U$ (moving upper surface)

The solution yields a Couette–Poiseuille velocity profile:

$$u(z) = \frac{z}{h} U - \frac{1}{2\mu} \frac{\partial p}{\partial x} z(h - z) \quad (3)$$

This profile accounts for both pressure-driven (Poiseuille) and shear-driven (Couette) flow.

Shear Stress Distribution in the Film

The shear stress $\tau(z)$ in the lubricant film is given by the Newtonian constitutive relation:

$$\tau(z) = \mu \frac{\partial u}{\partial z} \quad (4)$$

Using the Couette–Poiseuille velocity profile, this becomes:

$$\tau(z) = \mu \left(\frac{U}{h} - \frac{1}{2\mu} \frac{\partial p}{\partial x} (h - 2z) \right) \quad (5)$$

Derivation of Reynolds Equation

The classical Reynolds lubrication equation can be derived by taking the continuity equation over the thickness of the thin film, and replacing the velocity profile as derived from the simplified equations of the momentum equations. This equation mathematically models the distribution of pressure in the fluid film that includes contributions both to the pressure-driven (Poiseuille) flow stress and shear-driven (Couette) flow. The Reynolds equation is the foundation of the hydrodynamic lubrication theory, which allows calculating the key performance parameters of pressure distribution, load-carrying capability, shear forces, and shear stress in thin fluid films. Its development embodies the effect of significant parameters such as the thickness of the film, surface velocity, viscosity of the fluid, and boundary condition making it a flexible instrument in the examination of lubrication flows in bearings, sliders and other devices.

Unsteady (squeeze-film) Reynolds equation — adds transient effects

Equation

$$\frac{\partial}{\partial x} \left(\frac{h^3}{12\mu} \frac{\partial p}{\partial x} \right) = \frac{\partial(Uh)}{\partial x} + \frac{\partial h}{\partial t} \quad (6)$$

In simple geometries, analytical solutions of the Reynolds equation can give straightforward information on how a system behaves, whilst in more general surface profiles, non-linear film variations or operating conditions at a given point in time, non-linear methods, including a finite difference method, finite element method or finite volume method are usually used. The techniques of these computations enable engineers to accurately solve the Reynolds equation, make realistic predictions of lubrication behavior, and design mechanical systems to minimize the effects of friction, wear and enhance the reliability of working systems.

General mass-conserving cavitation (Elrod) form — handles cavitation/film rupture

Equation

$$\frac{\partial}{\partial x} \left(\frac{h^3}{12\mu} \frac{\partial p}{\partial x} \right) = \frac{\partial(Uh\theta)}{\partial x} + \frac{\partial h\theta}{\partial t}, 0 \leq \theta \leq 1 \quad (7)$$

Where $\theta(x, t)$ is the film fraction $\theta = 1$ full film, $\theta < 1$ cavitating

Solution Approaches:

The study of thin film lubrication often involves solving the Reynolds equation to determine pressure distribution, load-carrying capacity, and flow characteristics. Depending on the complexity of the geometry, fluid properties, and boundary conditions, different solution approaches can be adopted.

Analytical solutions are suitable for simple geometries with well-defined film thickness variations, providing exact expressions for pressure and velocity fields. However, in practical applications involving complex geometries, non-linear film profiles, non-Newtonian fluids, or transient operating conditions, analytical solutions are often intractable. In such cases, numerical methods are employed to obtain accurate and detailed predictions of lubrication performance. Both analytical and numerical approaches complement each other: analytical methods provide insights and validation benchmarks, while numerical techniques offer flexibility to handle real-world complexities and parametric studies.

General Steady-State Reynolds Equation (1D Form)

$$\frac{\partial}{\partial x} \left(\frac{h^3}{12\mu} \frac{\partial p}{\partial x} \right) = \frac{U}{2} \frac{\partial h}{\partial x} \quad (8)$$

This is the basic 1-dimensional steady Reynolds equation for incompressible Newtonian fluids.

Analytical Solutions

The analytical techniques are accurate in lubrication flows within simple geometries whereby the film thickness and the motion of the surfaces can be formulated using continuous functions. These methods provide explicit pressure distribution, velocity profiles, shear stress, and load-carrying capacity by expressing the governing equations directly (e.g. simplified Navier-Stokes or Reynolds equations). It is used as parallel plate bearings of constant thickness of film, inclined slider bearings of constant thickness of film over gradually varying thickness, and journal bearings of cylindrical geometries which generate sinusoidal or elliptical distributions of pressure. Analytical solutions are useful in gaining an insight into fundamental flow behavior, initial design and in the validation of numerical simulations:

- **Parallel Plate Bearings:** The fluid film has uniform thickness; pressure distribution is linear and can be obtained directly from the Reynolds equation.

10) Squeeze-film special-case (closed-form)

For simple oscillatory approach of parallel plates, a commonly-cited form:

- **Inclined Slider Bearings:** Film thickness varies linearly or nonlinearly along the sliding surface. Analytical integration of the Reynolds equation yields the pressure distribution and load-carrying capacity.
- **Journal Bearings:** Cylindrical geometry leads to a sinusoidal or elliptical pressure profile, which can be approximated using perturbation methods.

Numerical Methods

Analytical solutions are usually inadmissible due to complex geometries, non-linear variations in the thickness of the film, non-Newtonian lubricants and dynamic loading conditions. Numerical techniques are used in such instances:

- **Finite Difference Method (FDM):** Discretizes the Reynolds equation over a grid; widely used for slider and journal bearing simulations.

- **Finite Element Method (FEM):** Offers flexibility in handling irregular geometries and boundary conditions.
- **Finite Volume Method (FVM):** Conserves mass and momentum over control volumes; suitable for transient lubrication flows.

These calculation methods make it possible to investigate in details the distributions of pressure, velocity fields, and shear stresses in realistic conditions of operations. They can also be used to conduct parametric research into the effect of viscosity, surface geometry, temperature and the boundary conditions.

Results and Discussion

Analytical and numerical solutions of both models in the field of thin film lubrication give essential information about the dynamics of hydrodynamic flows, pressure distribution, and load bearing capacity of the lubricated device. The interpretation of these findings gives engineers an opportunity to assess the operation of bearings and other lubricated parts, optimize design variables and estimated the possible failure points. The discussion section below underlines the main results about the pressure distribution, impact of film thickness, influence of viscosity, and the strategies of the performance optimization of the bearings.

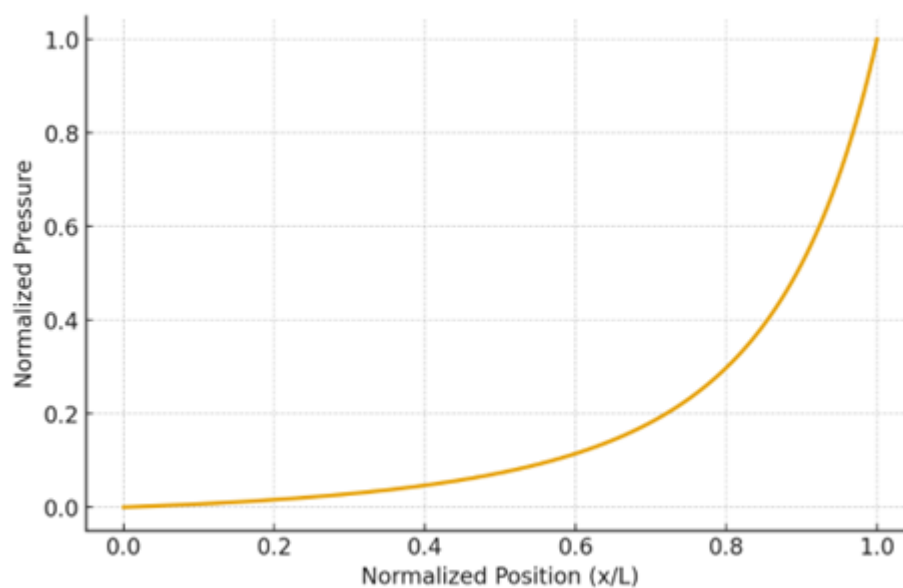


Figure 1: Graphical presentation of pressure distribution in an inclined slider bearing

Pressure Distribution

The hydrodynamic pressure in a thin film rises sharply in converging regions and decays in diverging regions. Maximum pressure occurs near the minimum film thickness. Analytical solutions for slider bearings confirm that small decreases in film thickness can significantly increase local pressure, enhance load-carrying capacity but potentially increase the risk of surface contact.

Effect of Film Thickness

Pressure and load-carrying capacity are inversely proportional to the cube of the film thickness ($p \propto 1/h^3$). Consequently, small variations in surface geometry or operational conditions have substantial impacts on system performance, highlighting the need for precise manufacturing and surface engineering.

Influence of Viscosity

Viscosity directly affects the magnitude of hydrodynamic pressure. Higher viscosity fluids generate larger pressures and improved load support, but may increase energy consumption due to viscous shear. Temperature-dependent viscosity introduces additional complexity, necessitating thermo-hydrodynamic analysis in high-speed or high-load applications.

Thermo-hydrodynamic (energy) equation — couples temperature to flow (T-dependent viscosity)

$$\rho c_p \left(u \frac{\partial T}{\partial x} + w \frac{\partial T}{\partial z} \right) = k \frac{\partial^2 T}{\partial z^2} + \mu \left(\frac{\partial u}{\partial z} \right)^2 \quad (10)$$

Temperature-Dependent Viscosity Relation

Lubricant viscosity decreases with temperature and is commonly modeled as:

$$\mu(T) = \mu_0 \exp[-\beta(T - T_0)] \quad (11)$$

Bearing Performance Optimization

The study emphasizes that proper selection of lubricant properties, surface geometry, and operating conditions is critical for maximizing bearing efficiency. Surface texturing and micro-structuring can further enhance hydrodynamic pressure peaks, reduce friction, and extend service life. In micro- and nano-scale applications, slip boundary conditions and scale-dependent fluid effects play a significant role in lubrication performance.

Conclusion

Thin film lubrication Mathematical modeling of thin film lubrication has become a general and fundamental model of analyzing and optimizing the hydrodynamic performance of mechanical systems, both at large scale, as in large mechanical bearings, and at small scale, as in microfluidic devices. Beginning with the general Navier-Stokes equations and using thin film approximations, the classical Reynolds equation is the foundation of the hydrodynamic lubrication theory, which allows the prediction of the distribution of pressure, load-carrying capacity, shear stress, and velocity profiles in thin fluid films. Analytical solutions provide useful information to simple geometries and numerical analysis can extrapolate this information to the complex surface, non-Newtonian fluids, and dynamically changing operating conditions. This paper has provided some of the most important results: the pressure distribution is very much dependent on the thickness of the film, the surface geometry, the load-carrying capacity is significantly reduced at smaller film thickness, and the viscosity of the fluid is important to the generation of pressure and the total efficiency of the system. Also, microstructuring and surface texturing are both effective methods of improving the performance of lubrication, and numerical modeling is essential in the analysis of nonlinear, real-world lubrication in practice. Though assumptions exist in simplified models, they are good first approximations which are needed in design, optimization or preliminary analysis.

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